PROBLEM 1:

Efficiency: is a description of how quickly your method of choosing is reaching the correct answer.

Stability: is a description of how robust a method is, The bisection method is a robust method and therefore stable.

Newton Raphson Preferable: this method both uses the function as well as the derivative of the function Therefore we can only use this method if we are able to compute the derivative of the given function.

Bisection Preferable: this method is always reliable and if we are not able to perform the derivative of the given function the bisection method is clearly the preferable one of these to algorithms.

Roots of functions of many variables: One can use the Newton Raphson method in multiple dimensions to solve this kind of problem. One can then systemize the components in a Jacobi matrix. One can now solve how much each component changes and then plug it back into the Newton Raphson method in multiple dimensions.

2. Finding the root of an equation:

Given:

f(x) = cosh(x)cos(x) -1

Root lies between the interval (4,5)

Solution:

Here cosh(x)cos(x)-1= 0

Let f(x) = cosh(x)cos(x)-1

Iteration 1:

Here f(4)= -18.85<0 and f(5)= 20.051>0

∴ Here root lies between 4 and 5

x0=(4+5)/2 = 4.5

f(x0) = f(4.5) = cosh(4.5)cos(4.5)-1 = -10.489<0

2nd iteration:

Here f(4.5)= -10.489<0 and f(5)= 20.051>0

∴ Here root lies between 4.5 and 5

x1=(4.5+5)/2 =4.75

f(x1) = f(4.75) = cosh(4.75)cos(4.75)-1 = 1.173>0

3rd iteration:

Here f(4.5)= -10.489<0 and f(4.75)= 1.173>0

∴Here root lies between 4.5 and 4.75

$x2=(4.5+4.75)/2 =4.625$

$f(x2) = f(4.625) = \cosh(4.625)\cos(4.625)-1= -5.452<0$

4th iteration:

Here $f(4.625)= -5.452<0$ and $f(4.75)= 1.173>0$

∴ Here root lies between 4.625 and 4.75

$x3=(4.625+4.75)/2 = 4.688$

$f(x3) = f(4.688) = \cosh(4.688)\cos(4.688)-1= -2.351<0$

5th iteration:

Here $f(4.688)=-2.351<0$ and $f(4.75)=1.173>0$

∴ Here root lies between 4.688 and 4.75

$x4=(4.688+4.75)/2 =4.719$

$f(x4) = f(4.719) = \cosh(4.719)\cos(4.719)-1= -0.644<0$

6th iteration:

Here $f(4.719)=-0.644<0$ and $f(4.75)=1.173>0$

∴ Here root lies between 4.719 and 4.75

$x5=(4.719+4.75)/2 =4.734$

$f(x5) = f(4.734) = \cosh(4.734)\cos(4.734)-1= 0.251>0$

7th iteration:

Here $f(4.719)=-0.644<0$ and $f(4.734)=0.251>0$

∴ Here root lies between 4.719 and 4.734

$x6=(4.719+4.734)/2. =4.727$

$f(x6) = f(4.727) = \cosh(4.727)\cos(4.727)-1= -0.2<0$

8th iteration:

Here $f(4.727)=-0.2<0$ and $f(4.734)=0.251>0$

∴ Here root lies between 4.727 and 4.734

$x7=(4.727+4.734)/2 =4.73$

$f(x7) = f(4.73) = \cosh(4.73)\cos(4.73)-1= 0.025>0$

9th iteration:

Here $f(4.727)=-0.2<0$ and $f(4.73)=0.025>0$

Here root lies between 4.727 and 4.73

$x8=(4.727+4.73)/2 = 4.729$

f(x8) = f(4.729) = cosh(4.729)cos(4.729)-1= -0.088<0

10th iteration:

Here f(4.729)=-0.088<0 and f(4.73)=0.025>0

∴ Here root lies between 4.729 and 4.73

x9=(4.729+4.73)/2 =44.729

f(x9) = f(4.729) = cosh(4.729)cos(4.729)-1= -0.032<0

11th iteration:

Here f(4.729)=-0.032<0 and f(4.73)=0.025>0

∴ Here root lies between 4.729 and 4.73

x10=(4.729+4.73)/2 =4.73

f(x10)= f(4.73) = cosh(4.73)cos(4.73)-1= -0.003<0

12th iteration:

Here f(4.73)=-0.003<0 and f(4.73)=0.025>0

∴ Here root lies between 4.73 and 4.73

x11=(4.73+4.73)/2 =4.73

f(x11)=f(4.73)=cosh(4.73)cos(4.73)-1= 0.011>0

13th iteration:

Here f(4.73)=-0.003<0 and f(4.73)=0.011>0

∴ Here root lies between 4.73 and 4.73

x12=(4.73+4.73)/2 =4.73

f(x12) = f(4.73)= cosh(4.73)cos(4.73)-1= 0.004>0

14th iteration:

Here f(4.73)=-0.003<0 and f(4.73)=0.004>0

∴ Here root lies between 4.73 and 4.73

x13=(4.73+4.73)/2 = 4.73

f(x13)= f(4.73)= cosh(4.73)cos(4.73)-1= 0>0

Approximate root of the equation cosh(x)cos(x)-1=0 using Bisection method is 4.73

3. **Richardson Extrapolation:**

Richardson Extrapolation is one the technique of numerical analysis which is used to estimate the error in the solution by solving the problem with two different grid sizes, provided a functional form of the solution is known.

4. Finding the approximation of f'(2.37) from the following data:

Given:

The value of table for x and y

x	=2.36,	2.37,	2.38,	2.39
y=	0.59,	0.63	, 0.67,	0.71

Solution:

Lagrange's Interpolating Polynomial

Lagrange's Interpolation formula is $f(x)=((x-x1)(x-x2)(x-x3)/ (x0-x1)(x0-x2)(x0-x3)) \times y0 +$

$((x-x0)(x-x2)(x-x3)/(x1-x0)(x1-x2)(x1-x3)) \times y1 +$

$((x-x0)(x-x1)(x-x3)/(x2-x0)(x2-x1)(x2-x3)) \times y2 +$

$((x-x0)(x-x1)(x-x2)/(x3-x0)(x3-x1)(x3-x2)) \times y3$

$f(x)=((x-2.37)(x-2.38)(x-2.39)/(2.36-2.37)(2.36-2.38)(2.36-2.39)) \times 0.59 +$

$((x-2.36)(x-2.38)(x-2.39)/ (2.37-2.36)(2.37-2.38)(2.37-2.39)) \times 0.63 +$

$((x-2.36)(x-2.37)(x-2.39)/$

$(2.38-2.36)(2.38-2.37)(2.38-2.39)) \times 0.67 +$

$((x-2.36)(x-2.37)(x-2.38)/$

$(2.39-2.36)(2.39-2.37)(2.39-2.38)) \times 0.71$

$f(x)=((x-2.37)(x-2.38)(x-2.39)/$

$(-0.01)(-0.02)(-0.03)) \times 0.59 +((x-2.36)(x-2.38)(x-2.39)/(0.01)(0)(-0.02)) \times 0.63 +((x-2.36)(x-2.37)(x-2.39)/(0.02)(0.01)(-0.01)) \times 0.67 +((x-2.36)(x-2.37)(x-2.38)/(0.03)(0.02)(0.01)) \times 0.71$

$f(x)=(x^3-7.14x^2+16.99x-13.48)/0 \times 0.59+(x^3-7.13x^2+16.95x-13.42)/0$

$\times 0.63 +(x^3-7.12x^2+16.9x-13.37)/0$

$\times 0.67 +(x^3-7.11x^2+16.85x-13.31)/0 \times 0.71$

$f(x)=(x^3-7.14x^2+16.99x-13.48) \times -97766.67 + (x^3-7.13x^2+16.95x-13.42) \times 314450 +(x^3-7.12x^2+16.9x-13.37) \times -335500 +(x^3-7.11x^2+16.85x-13.31) \times 118816.67$

$f(x)=(-97766.67x^3 + 698054x^2 -1661358.74x+1317995.76) +$

$(314450x^3-2242028.5x^2 + 5328481.03x -4221224.6) +(-335500x^3 +2388760x^2 - 5669245.45x+4484879.45) + (118816.67x^3-844786.5x^2 +2002132.12x-1581665.6)$

$f(x)=-(-x^2)+8.96x-14.99$

$f'(x) = -2x +8.96$

Substitute x= 2.37

$f'(2.37)= - 2(2.37) + 8.96$

$f'(2.37) = -4.74+8.96$

Thus $f'(2.37)= 4.22$

5. **Romberg Integration:**

Romberg Integration is one of the interpolation techniques that gives us a sequence approximation solution to an integral. This technique gives us a better approximation.

Gaussian Quadrature:

In numerical analysis Gaussian quadrature is a rule to approximate the definite integral of a function.

Comparison of Romberg Integration and Gaussian Quadrature:

- Gaussian Quadrature method is much more accurate than Romberg Integration method.
- For corresponding error limit Gaussian Quadrature method needs fewer integration points. So, it requires much less computer time.

Problem No 6

$$\int_{-\infty}^{\infty} e^{-\frac{x^4}{(1+x^2)}}$$

$$f(x) = e^{-\frac{x^4}{(1+x^2)}}$$

x	-8	-6	-4	-2	0	2	-4	-6	-8
$f(x)$	4.29×10^{-28}	6.14×10^{-16}	2.88×10^{-7}	0.040	1	0.040	2.88×10^{-7}	6.14×10^{-16}	4.29×10^{-28}

Let approximate the function from approximatly $x = 3$ to ∞ & $x = -3$ to $x \to -\infty$

Therefore now the integration limit ± 2.5

x	x_0 -3	x_1 -2.25	x_2 -1.5	x_3 -0.75	x_4 0	x_5 0.75	x_6 1.5	x_7 2.25	x_8 3
$f(x)$	3.035×10^{-4}	0.0146	0.2106	0.8167	1	0.8167	0.2106	0.0146	3.035×10^{-4}

By Romberg Integration

$$\begin{pmatrix} R_{1,1} & & & \\ R_{2,1} & R_{2,2} & & \\ R_{3,1} & R_{3,2} & R_{3,3} & \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} \end{pmatrix}$$

$h = b - a$

$= 3 - (-3) = 6$

$\Rightarrow h = 6$

$$R_{1,1} = h\left(\frac{f(x_0)}{2} + \frac{f(x_8)}{2}\right)$$

$$= 6\left(\frac{3.035 \times 10^{-4}}{2} + \frac{3.035 \times 10^{-4}}{2}\right)$$

$R_{1,1} = 1.82 \times 10^{-3}$

$$R_{2,1} = \frac{h}{2}\left(\frac{f(x_0)}{2} + f(x_4) + \frac{f(x_8)}{2}\right)$$

$$= \frac{6}{2}(1.0003) = 3.0091$$

$$R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{3}$$

$R_{2,2} = 4.00607$

$$R_{3,1} = \frac{h}{4}\left(\frac{f(x_0)}{2} + f(x_4) + f(x_6) + \frac{f(x_8)}{2}\right)$$

By putting the values of x_0, x_4, x_6 & x_8

$R_{3,1} = 2.132$

$$R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{3} = 1.842$$

$$R_{3,3} = \frac{16R_{3,2} - R_{2,2}}{15} = 1.698$$

$$R_{4,1} = \frac{h}{8}\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + \frac{f(x_8)}{2}\right)$$

= By putting the values of $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$ & x_8 and solving we get

$$R_{4,1} = 2.313$$

$$R_{4,2} = \frac{4R_{4,1} - R_{3,1}}{3} = 2.373$$

$$R_{4,3} = \frac{16R_{4,2} - R_{3,2}}{15} = 2.408$$

$$R_{4,4} = \frac{64R_{4,3} - R_{3,1}}{63} = 2.419$$

$$\text{Romberg} = \begin{array}{cccc} h & h^2 & h^4 & h^8 \\ \end{array}$$
$$\begin{bmatrix} 1.821\times10^{-3} & \cancel{4.006} & & \\ 3.0091 & 4.006 & & \\ 2.132 & 1.842 & 1.69 & \\ 2.313 & 2.37 & 2.408 & 2.419 \end{bmatrix}$$

$$R_{4,4} = I_{exact} + h^8$$

$$\Rightarrow I_{exact} = R_{4,4} - h^8$$

The uncertainty $h^8 = 0.75^8 \approx 0.1001$

$$I_{exact} = 2.319$$

PROBLEM 7

Stiffness: of the solution of a ODE is varying slowly and closely Solutions are varying much more means that the ODE is stiff. As a result of this we must take small steps to require a result which we are satisfied with.

Adaptive step size: if we have a function which is not very volatile, we should increase our stepsister If we don't do this we are doing a lot more computations than we actually need to do But it is difficult to choose the correct stepsister at all times this is why we use the adaptive step size method which estimates the truncation errors and adjusts the step size accordingly.

Prob. 8

estimate: $y(0.5)$ given the ordinary

$y'(x) = \sin(y(x))$

with the initial value $y(0)=1$

By implementing the Runge Kutta

$y(0) = 1$

$x_0 = 0, \quad y_0 = 1$

$k_0 = hF(x, y)$

$= 0.1 \cdot \sin(y(0))$

$= 0.1 \cdot \sin(1)$

$k_1 = h \cdot F(x + \frac{h}{2}, y + \frac{k_0}{2})$

$k_1 = 0.1 \sin(y(0.05))$

$\Rightarrow k_1 = 0.1 \sin(1.042)$

$k_2 = h \cdot F(x + \frac{h}{2}, y + \frac{k_1}{2})$

$= 0.1 \sin(y(0.05))$

$k_2 = 0.1 \sin(1.043)$

$k_3 = h \cdot F(x+h, y+k_2)$

$= 0.1 \sin(y(0.1))$

$\Rightarrow k_3 = 0.1 \sin(1.086)$

$y(x+h) = y(x) + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$

$y(0.1) = 1.086$

for $x = 0$ & $y = 1.086$

$k_0 = h \cdot F(x, y)$

$= 0.1 \sin(1.086)$

$k_1 = h \cdot F(x + \frac{h}{2}, y + \frac{k_0}{2})$

$= 0.1 \sin(1.13)$

$k_2 = h \cdot F(x + \frac{h}{2}, y + \frac{k_1}{2})$

$= 0.1 \sin(1.131)$

$k_2 = 0.1 \sin(1.131)$

$k_3 = hF(x+h, y+k_2)$

$= 0.1 \sin(1.176)$

$y(0.2) = 1.176$

Now $x = 0.2$ and $y = 1.176$

$k_0 = h \cdot F(x, y) = 0.1 \cdot \sin(1.176)$

$k_1 = h \cdot F(x + \frac{h}{2}, y + \frac{k_0}{2})$

$k_1 = 0.1 \sin(1.22)$

$k_2 = h \cdot F(x + \frac{h}{2}, y + \frac{k_1}{2})$

$k_2 = 0.1 \sin(1.27)$

$k_3 = h \cdot F(x+h, y+k_2)$

$= 0.1 \sin(1.27)$

$y(0,3) = 1.27$

Now

$x = 0.3$ & $y = 1.27$

$k_0 = h \cdot F(x,y)$

$= 0.1 \sin(1.27)$

$k_1 = h \cdot F\left(x+\frac{h}{2}, y+\frac{k_0}{2}\right)$

$k_1 = 0.1 \sin(1.318)$

$k_2 = h \cdot F\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$

$k_2 = 0.1 \sin(1.319)$

$k_3 = h \cdot F(x+h, y+k_2)$

$= 0.1 \cdot \sin(1.367)$

$y(0.4) = 1.367$

Now $x = 0.4$ and $y = 1.367$

$k_0 = h \cdot F(x,y)$

$= 0.1 \sin(1.367)$

$k_1 = h \cdot F\left(x+\frac{h}{2}, y+\frac{k_0}{2}\right)$

$= 0.1 \sin(1.416)$

$k_2 = h \cdot F\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$

$= 0.1 \sin(1.417)$

$k_3 = h \cdot F(x+h, y+k_2)$

$= 0.1 \sin(1.466)$

$y(0.5) = 1.466$

Problem 10

$0 \le x \le 4$

Boundary Conditions:-

$$y(0)=0 \quad \text{and} \quad y'(1)=4$$

$$y''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$y'(x_i) = \frac{y_{i+1} - y_{i-1}}{2h}$$

x_i	0	1	2	3	4
y_i	0	y_1	y_2	y_3	y_4

$h = 1$

$y'(1) = 4$

$$y'(x_1) = \frac{y_2 - y_0}{2(h)} = 4$$

$y_2 - 0 = 4(2) \Rightarrow y_2 = 8$

x_i	0	1	2	3	4
y_i	0	y_1	8	y_3	y_4

$y''(x) = (2+x) \cdot y(x)$

$\Rightarrow y''(x_i) = (2 + x_i) \cdot y_i$

$y'' = 2y_i + x_i y_i$

$$y''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$i = 1$

$$\frac{y_0 - 2y_1 + y_2}{h^2} = 2y_1 + 1(y_1) \quad \cancel{y''(x_1) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}}$$

$$y_0 - 2y_1 + y_2 = 3y_1$$

$$\Rightarrow \; y_2 = 3y_1 + 2y_1$$

$$y_2 = 5y_1 \longrightarrow (1)$$

$i = 2$

$$\frac{y_1 - 2y_2 + y_3}{h^2} = 2y_2 + 2(y_2)$$

$$y_1 + y_3 = 6y_2$$

$$y_3 = 6y_2 - y_1 \quad \text{---} \quad (2)$$

$i = 3$

$$\frac{y_2 - 2y_3 + y_4}{h^2} = 2y_3 + 3y_3$$

$$y_2 + y_4 = 7y_3$$

$$y_4 = 7y_3 - y_2 \quad \text{---} \quad (3)$$

As we know $y_2 = 8$

$y_2 = 5y_1$

$\Rightarrow 8 = 5y_1 \Rightarrow y = \frac{8}{5} = 1.6$

$y_3 = 6y_2 - y_1$

$= 6(8) - 1.6$

$y_3 = 46.4$

$y_4 = 7y_3 - y_2$

$= 7(46.4) - 8$

$y_4 = 316.8$

x_i	0	1	2	3	4
y_i	0	1.6	8	46.4	316.8